# Online Appendix for "On the Efficiency of Stable Matchings in Large Markets"

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**Abstract**

In this Online Appendix, we provide an alternative proof for asymptotic average efficiency, a tighter convergence result, and an additional bound on the maximal efficiency, for markets with fully idiosyncratic preferences. We also consider environments in which the support of individual match utilities grows with the market's size. We characterize the speeds at which these bounds can increase and still guarantee that stable outcomes will be asymptotically average efficient in the various market settings considered in the paper. Last, we discuss the robustness of our results to the possibility of some individuals being unacceptable in the market.

## 1 Fully Idiosyncratic Preferences

When preferences are completely idiosyncratic, there is a short proof of asymptotic average efficiency and a tighter characterization of the corresponding convergence speed than that offered by our general results for hybrid models. Formally, we have the following:

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**Proposition 1 (Fully Idiosyncratic Preferences)** *With fully idiosyncratic preferences,*
*1. Stable matchings are asymptotically average efficient:*

$$\lim_{n\to\infty} \frac{S_n^f}{n} = \lim_{n\to\infty} \frac{S_n^w}{n} = 1,$$

*and, in particular,* $\lim_{n\to\infty} \frac{S_n}{2n} = 1$.
*2. When utilities are drawn from the uniform distribution,*

$$\lim_{n\to\infty} \left(1 - \frac{S_n^f}{n}\right) \log n = \lim_{n\to\infty} \left(1 - \frac{S_n^w}{n}\right) \log n = 1.$$

**Proof of Proposition 1:**

1. We assume that all utilities are drawn from the uniform distribution over $[0,1]$. A similar technique to that used in the proof regarding fully aligned preferences presented in the body of the paper can be used to generalize the result to arbitrary continuous distributions with full supports.

For any realized market, we denote by $\mu^w$ the firm-pessimal, or worker-optimal, stable matching. We want to show that

$$\frac{S_n^f}{n} = \frac{\mathbb{E}\left[\sum_{i=1}^n u_{i\mu^w(i)}\right]}{n} \to 1.$$

We consider a two-step procedure for generating idiosyncratic preferences. First, an ordinal preference profile $\succ$ is drawn from the uniform distribution over the set of all possible preference profiles. That is, each firm has a preference list that is drawn uniformly from the set of permutations of $n$ workers. Each worker's preference is similarly generated. Given a realization of $\succ$, cardinal utilities for each agent are generated as follows. $n$ numbers are drawn from the uniform distribution on $[0,1]$. The highest number is then the match utility resulting from a match with the agent's most preferred partner, the second highest is the match utility resulting from matching with the second preferred partner, etc.

This two-step procedure then implies:

$$S_n^f = \mathbb{E}\left[\sum_{i=1}^n u_{i\mu^w(i)}\right] = \mathbb{E}_\succ\left[\mathbb{E}_{u|\succ}\left[\sum_{i=1}^n u_{i\mu^w(i)}| \succ\right]\right].$$

Let $R_i^f(\mu^w)$ denote the rank number of firm $f_i$'s worker-optimal stable matching partner. If the firm is matched to its most preferred worker, the rank number is 1. Also, let $u_{[k;n]}$ be $k$'th highest value from a sample of size $n$ from the uniform distribution on $[0,1]$.

As the preference profile determines the rank number of the worker-optimal stable matching partners, and since the expected $k$'th highest value corresponding to the uniform distribution is given by $1-\frac{k}{n+1}$, we can write

$$\begin{aligned} S_n^f &= \mathbb{E}_\succ\left[\mathbb{E}_{u|\succ}\left[\sum_{i=1}^n u_{i\mu^w(i)}| \succ\right]\right] = \mathbb{E}_\succ\left[\mathbb{E}_{u|\succ}\left[\sum_{i=1}^n u_{[R_i^f(\mu^W);n]}| \succ\right]\right] \\ &= \mathbb{E}_\succ\left[\sum_{i=1}^n\left(1-\frac{R_i^f(\mu^w)}{n+1}\right)\right] = n - \frac{\mathbb{E}_\succ\left[\sum_{i=1}^n R_i^f(\mu^w)\right]}{n+1}. \end{aligned}$$

Theorem 2 in Pittel (1989) proves that

$$\frac{\left(\sum_{i=1}^n R_i^f(\mu^w)\right)\log n}{n^2} \xrightarrow{p} 1.$$

It is immediate that $\frac{\sum_{i=1}^n R_i^f(\mu^w)}{n^2} \xrightarrow{p} 0$. As $\frac{\sum_{i=1}^n R_i^f(\mu^w)}{n^2}$ is bounded above by 1 with probability 1, by Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty}\frac{\mathbb{E}\left[\sum_{i=1}^n R_i^f(\mu^w)\right]}{n^2} = \lim_{n\to\infty}\frac{\mathbb{E}_\succ\left[\sum_{i=1}^n R_i^f(\mu^w)\right]}{n(n+1)} = 0.$$

2. We now show that

$$\lim_{n\to\infty}\left(1-\frac{S_n^f}{n}\right)\log n = \lim_{n\to\infty}\frac{\left(\mathbb{E}_\succ\left[\sum_{i=1}^n R_i^f(\mu^w)\right]\right)\log n}{n(n+1)} = 1.$$

We use two results from which Pittel (1989) obtains its main Theorem.

First, Equation (4.4) in Pittel (1989) assures that for any small $\rho > 0$ and $\delta \in (0, e^{\rho} - 1)$,

$$P\left(\sum_{i=1}^{n} R_i^f(\mu^w) \leq \frac{n^2}{\log n}\left(1 + \frac{\log\log n + \rho}{\log n}\right)\right) \geq 1 - O(n^{-\delta}).$$

Thus,

$$\frac{\mathbb{E}\left[\sum_{i=1}^{n} R_i^f(\mu^w)\right]}{n^2} \leq (1 - O(n^{-\delta}))\left(\frac{1}{\log n}\right)\left(1 + \frac{\log\log n + \rho}{\log n}\right) + O(n^{-\delta}),$$

which implies that

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[\sum_{i=1}^{n} R_i^f(\mu^w)\right]\log n}{n^2} \leq 1.$$

In addition, the result on page 545 of Pittel (1989) assures that

$$\mathbb{E}\left[\sum_{i=1}^{n} R_i^f(\mu^w)\right] \geq \frac{n^2}{\log n}\left(1 + O\left(\frac{1}{\log n}\right)\right),$$

which implies that

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[\sum_{i=1}^{n} R_i^f(\mu^w)\right]\log n}{n^2} \geq 1.$$

■

In fact, as mentioned in the text, we can also show that, when all utilities are drawn from the uniform distribution over $[0, 1]$, $\lim_{n\to\infty} \frac{2n - E_n}{\sqrt{n}} \geq \sqrt{\frac{\pi}{2}}$. Thus, the difference between $E_n$ and $2n$ is of the order of $\sqrt{n}$.

**Proposition 2 (Fully Idiosyncratic Preferences - Maximal Efficiency)** *With fully idiosyncratic preferences,*

$$\lim_{n\to\infty} \frac{2n - E_n}{\sqrt{n}} \geq \sqrt{\frac{\pi}{2}}.$$

**Proof of Proposition 2:**

Let $\tilde{u}_{ij} \equiv \frac{u^f_{ij}+u^w_{ij}}{2}$. Notice that

$$P\left(\tilde{u}_{ij} \le x\right) = \begin{cases} 2x^2 & \text{if } x \le 1/2 \\ 1-2(1-x)^2 & \text{if } x \ge 1/2. \end{cases}$$

Consider now a market with aligned preferences with utilities $v_{ij}$ drawn from a distribution

$$P\left(v_{ij} \le x\right) = \begin{cases} 0 & \text{if } x \le 1-1/\sqrt{2} \\ 1-2(1-x)^2 & \text{if } 1-1/\sqrt{2} \le x \le 1. \end{cases}$$

Then, $v_{ij}$ first order stochastically dominates $\tilde{u}_{ij}$.

We denote by $E^v_n$ the maximal aggregate efficiency from the market with aligned preferences $v_{ij}$ and show that $\lim_{n\to\infty} \frac{2n-E^v_n}{\sqrt{n}} \ge \sqrt{\frac{\pi}{2}}$.

We use a construction similar to that of Lazarus (1993) and Goemans and Kodialam (1993). Define $c_{ij} \equiv 1 - v_{ij}$, which represents the 'cost' of matching $f_i$ and $w_j$ to each. Note that $2n - E^v_n$ is the expected value of the following assignment problem:

$$\begin{aligned} &\min_{\{x_{ij}\}_{i,j}} 2\left(\textstyle\sum_{i\in F}\sum_{j\in W} c_{ij}x_{ij}\right) \\ &\text{subject to } \textstyle\sum_{j\in W} x_{ij} = 1, \sum_{i\in F} x_{ij} = 1, x_{ij} \ge 0. \end{aligned}$$

For each realization of $\{v_{ij}\}_{i,j}$, the value of the above linear program is equal to

$$\begin{aligned} &\max_{\{y_i\}_{i\in F},\{z_j\}_{j\in W}} 2\left(\textstyle\sum_{i\in F} y_i + \sum_{j\in W} z_j\right) \\ &\text{subject to } y_i + z_j \le c_{ij}. \end{aligned}$$

We consider a feasible solution for the above linear program. For realizations of $c_{ij}$, each firm $f_i$ selects a worker corresponding to $r_i = \arg\min_j c_{ij}$. Let $y_i = c_{ir_i}$ and $z_j = 0$ for every $w_j \in W$.

We compute the expected value of a feasible solution:

$$2\mathbb{E}\left[\sum_{i\in F} y_i + \sum_{j\in W} z_j\right] = 2n\mathbb{E}[y_1]$$

This is a lower bound on the expected value of the solution of the dual linear program, so it is a lower bound on $2n - E_n^v$.

Note that $\mathbb{E}[y_1]$ is the expected value of the lowest value from $n$ samples $\{c_{1j}\}_{w_j\in W}$. Thus, $y_1$ has a cumulative distribution function:

$$\begin{aligned} G(x) &= Pr(y_1 \le x) = 1 - (Pr(c_{1j} > x))^n = 1 - [F^v(1-x)]^n \\ &= 1 - (1-2x^2)^n \quad \text{for } x \le 1/\sqrt{2}. \end{aligned}$$

Therefore,

$$\mathbb{E}[y_1] = \int_0^{1/\sqrt{2}} 1 - G(x)dx = \int_0^{1/\sqrt{2}} (1-2x^2)^n dx = \frac{\sqrt{\pi}\Gamma(n+1)}{2\sqrt{2}\Gamma(n+\frac{3}{2})},$$

and $\lim_{n\to\infty} \sqrt{n}E[y_1] = \frac{1}{2}\sqrt{\frac{\pi}{2}}$. ■

# 2 Fully Assortative Preferences on One Side

We consider here a polar case in which all workers agree on the ranking and the valuations of the firms, while firms have independent evaluations of workers. Formally, we assume that $u_{ij}^f$ are independently distributed according to the uniform distribution on $[0, 1]$. Unlike before, we assume that for any $w_j$ and $w_{j'}$, $u_i^w \equiv u_{ij}^w = u_{ij'}^w$ for any firm $f_i$. We assume that worker's utilities are also independently distributed according to the uniform distribution on $[0, 1]$.

Generically, realized markets will entail a unique stable matching. Indeed, the unique stable matching is assortative: The most desirable firm matches with her highest ranked worker (indeed, they are each other's favorite partner in the market); the second most desirable firm then matches with her highest-ranked worker of the remaining $n-1$; and so on.

We use the same notation as before for the average and aggregate efficiency experienced by firms and workers under the (generically) unique stable matching. Since match utilities are determined independently in the stable matching and, in fact, in any market matching, one of the workers will receive a utility corresponding to the highest entry of $n$ samples from the uniform distribution (from matching with the highest ranked firm), one will receive a utility corresponding to the next highest entry, etc. Therefore,

$$S_n^w = \frac{n}{n+1} + \frac{n-1}{n+1} + \frac{n-2}{n+1} + ... + \frac{1}{n} = \frac{n}{2}.$$

For firms, the most desirable firm's expected utility is the expectation of the highest entry of $n$ uniform random variables (corresponding to the $n$ workers), i.e., $\frac{n}{n+1}$. The second most preferred firm's expected utility is the expectation of the highest entry of $n-1$ uniform variables, $\frac{n-1}{n}$, etc. Therefore,

$$S_n^f = \frac{n}{n+1} + \frac{n-1}{n} + ... + \frac{1}{2} = n - \sum_{k=1}^{n} \frac{1}{k+1}.$$

Notice that

$$\log(n+2) - \ln 2 = \int_2^{n+2} \frac{1}{x} dx \le \sum_{k=1}^{n} \frac{1}{k+1} \le \int_1^{n+1} \frac{1}{x} dx = \log(n+1).$$

Now, recall that any matching involves the same expected payoff for workers (of $\frac{1}{2}$). It follows that stability achieves the first best in terms of average or aggregate efficiency for workers. Moreover, the average efficiency for firms converges to 1.[1] Formally,

**Proposition 3 (One-sided Assortative Preferences - Asymptotic Average Efficiency)** *For all $n$,*

$$\frac{S_n^w}{n} = \frac{1}{2} \quad \textit{and} \quad \frac{\log(n+2) - \log 2}{n} \le 1 - \frac{S_n^f}{n} \le \frac{\log(n+1)}{n}.$$

*In particular, $\lim_{n\to\infty} \frac{S_n^f}{n} = 1$ and stable matchings are asymptotically average efficient.*

[1] Naturally, had we assumed that firms share the same evaluations of workers (so that the market was ex-ante symmetric), the average efficiency both firms and workers experience would be $\frac{1}{2}$ in any market matching, in particular the unique stable one.

The speed of convergence in this setting is of the order of $\frac{\log n}{n}$, as in the case of fully aligned (and uniform) utilities.

The analysis of aggregate efficiency with fully assortative preferences as studied here follows directly from our analysis of markets with fully aligned preferences (Proposition 3 of the paper). Indeed, in these markets, in both the stable and optimal matchings, workers' average efficiency is $1/2$. The maximal average efficiency is driven by finding a matching that maximizes the utilitarian welfare firms receive, which is derived as for the case of aligned markets. Furthermore, from Proposition 3 here, the speeds with which the average efficiency of stable matchings converges is comparable across these two types of markets. The results in the paper then suggest that for assortative preferences, the aggregate efficiency loss from implementation of stable matchings is of the order of $\log n$.

# 3 Unbounded Supports

Our assumption that utilities are drawn from bounded supports plays a role in our analysis throughout the paper. Nonetheless, the results still hold for certain relaxations of the bounded support assumption. In this section we first study markets in which utilities are drawn from uniform distributions that have increasing supports as the market size grows. While the characterization of all unbounded distributions and preference structures that assure asymptotic average efficiency is beyond the scope of this paper, we also illustrate the interesting case of aligned preferences drawn from an (unbounded) exponential distribution, for which the average efficiency of stable matchings has a closed-form characterization. In this case also, stable matchings are asymptotically average efficient.

## 3.1 Uniform Distributions with Increasing Supports

Let $\{a_n\}$ be an arbitrary increasing sequence. We consider a sequence of markets such that, for each market with $n$ firms and $n$ workers, the marginal distribution of each match utility $u_{ij}$ is uniform over $[0, a_n]$. We now characterize the sequences $\{a_n\}$ for which stable matchings are asymptotically average efficient.

For fully aligned preferences, we continue to use our notation of $S_n$ as the aggregate efficiency of stable matchings in a market with $n$ firms and workers. We let $S_n^{[0,1]}$ denote the aggregate efficiency of stable matchings in the same-sized market, where the marginal distribution of match utilities is uniform over $[0,1]$. In other markets, where there may exist multiple stable matchings or asymmetry between the two sides of the market, we denote by $S_n^f$ and $S_n^w$ the aggregate efficiency (i.e., the expected utilitarian welfare from the worst-case stable matching) for firms and workers, respectively. $S_n^{f,[0,1]}$ and $S_n^{w,[0,1]}$ are then defined analogously.

It follows that in markets with a unique stable matching:

$$\frac{S_n}{2n} = \frac{S_n^{[0,1]}}{2n} a_n,$$

and in markets with possible multiplicity of stable matchings:

$$\frac{S_n^f}{2n} = \frac{S_n^{f,[0,1]}}{2n} a_n, \quad \text{and} \quad \frac{S_n^w}{2n} = \frac{S_n^{w,[0,1]}}{2n} a_n.$$

**Fully Aligned Preferences** We consider here the case of $\alpha = 0$ of Proposition 2 in the paper. From part 1 of that proposition, for any $n \geq 3$,

$$\frac{1}{2} \frac{\log n}{n} \leq 1 - \frac{S_n^{[0,1]}}{2n} \leq \frac{\log n}{n}.$$

Therefore,

$$\lim_{n \to \infty} \left( a_n - \frac{S_n}{2n} \right) = \lim_{n \to \infty} a_n \left( 1 - \frac{S_n^{[0,1]}}{2n} \right) = 0,$$

if and only if

$$\lim_{n \to \infty} a_n \frac{\log n}{n} = 0.$$

As long as $a_n$ increases slower than $\frac{n}{\log n}$, stable matchings for fully aligned preferences with utilities drawn from the uniform distribution on $[0, a_n]$ are asymptotically average efficient.

**Fully Idiosyncratic Preferences** For the case of $\alpha = 1$ in Proposition 2 in the paper, we can use the results in Proposition 1 of this Online Appendix. For $x = f, w$,

$$\lim_{n\to\infty}\left(1 - \frac{S_n^{x,[0,1]}}{n}\right)\log n = 1.$$

Therefore, for $x = f, w$,

$$\lim_{n\to\infty}\left(a_n - \frac{S_n^x}{n}\right) = \lim_{n\to\infty} a_n\left(1 - \frac{S_n^{x,[0,1]}}{n}\right) = 0,$$

which occurs if and only if

$$\lim_{n\to\infty}\frac{a_n}{\log n} = 0.$$

That is, as long as $a_n$ increases slower than $\log n$, stable matchings for fully idiosyncratic preferences with utilities drawn uniformly from $[0, a_n]$ are asymptotically average efficient.

**Assortative Preferences** Whenever preferences are fully assortative—workers all share the same evaluation of firms and firms all share the same evaluation of workers—any matching that does not leave agents unmatched is equally utilitarian efficient, for any realized common values. In particular, the supports of the distribution play no role.

Consider then the model in Section 2 here: workers all share the same evaluation of firms with utilities determined uniformly, while firms have independent evaluations of workers. Formally, we assume that $u_{ij}^f$ are independently and identically distributed according to the uniform distribution over $[0, a_n]$. We assume that for any $w_j$ and $w_{j'}$, $u_i^w \equiv u_{ij}^w = u_{ij'}^w$ for all firms $f_i$, where $(u_i^w)_i$ are also drawn independently and uniformly over $[0, a_n]$. In the unique stable matching, workers get a random draw of utilities, which is average and aggregate efficient, as discussed in the paper. We focus on the firms and use notation as above.

From Proposition 3, for every $n$,

$$\frac{\log(n+2) - \log 2}{n} \leq 1 - \frac{S_n^{f,[0,1]}}{n} \leq \frac{\log n}{n}.$$

Therefore, asymptotic average efficiency translates into

$$\lim_{n\to\infty}\left(a_n - \frac{S_n^f}{n}\right) = \lim_{n\to\infty} a_n\left(1 - \frac{S_n^{f,[0,1]}}{n}\right) = 0,$$

which occurs if and only if

$$\lim_{n\to\infty} a_n \frac{\log n}{n} = 0.$$

That is, much like in the case of aligned preferences, stable matchings are asymptotically average efficient as long as $a_n$ diverges at a speed lower than $\frac{n}{\log n}$.

**Linear Model of Aligned Preferences with Idiosyncratic Shocks** From Proposition 2 in the body of the paper, for any $\alpha \in (0,1)$, for $x = f, w$,

$$\limsup_{n\to\infty}\left(1 - \frac{S_n^{x,[0,1]}}{n}\right)\log n \leq 2.$$

Therefore,

$$\lim_{n\to\infty}\left(a_n - \frac{S_n^x}{n}\right) = \lim_{n\to\infty} a_n\left(1 - \frac{S_n^{x,[0,1]}}{n}\right) = 0,$$

which occurs if

$$\lim_{n\to\infty}\frac{a_n}{\log n} = 0$$

so that asymptotically average efficient stable matchings are guaranteed as long as $a_n$ diverges at a speed lower than $\log n$.

**Linear Model of Assortative Preferences with Idiosyncratic Shocks** We consider the model underlying Proposition 5 in the body of the paper. For $\beta \in (0,1)$, for $x = f, w$,

$$(1-\beta)\frac{1}{2} + \beta - \frac{S_n^{x,[0,1]}}{n} = O(n^{-1/4}).$$

The maximal conceivable efficiency when all preference components are uniformly distributed over $[0, a_n]$ is:

$$a_n \left[ (1-\beta)\frac{1}{2} + \beta \right].$$

As long as $\lim_{n\to\infty} a_n n^{-1/4} = 0$,

$$\lim_{n\to\infty} \left( a_n \left( (1-\beta)\frac{1}{2} + \beta \right) - \frac{S_n^x}{n} \right) = \lim_{n\to\infty} a_n \left( (1-\beta)\frac{1}{2} + \beta - \frac{S_n^{x,[0,1]}}{n} \right) = 0.$$

Notice that the results can be extended to arbitrary distributions with expanding supports that are stochastically dominated by some sequence of uniform distributions with supports that satisfy the condition for asymptotically average efficient stable matchings. Nonetheless, as mentioned in the body of the paper, when supports expand rapidly enough, stable matchings may no longer be asymptotically average efficient. For instance, suppose $a_n = n$. The analysis of average efficiency is then tantamount to that of aggregate efficiency when supports are bounded and fixed. As we show in the body of the paper, the aggregate efficiency loss does not vanish asymptotically in the settings we consider.

## 3.2 Exponentially Distributed Utilities

Consider aligned preferences in which $u_{ij} = -x$, where $x \sim \exp(1)$, drawn independently for each $i, j$, so that utilities range form 0 to $-\infty$. The exponential distribution has an appealing feature: we can calculate precisely the average or aggregate efficiency generated by stable matchings.

Indeed, recall our construction of stable matchings in Section 3.3. We first find the pair corresponding to the highest realized match utility. This pair must be matched in the stable matching. The expected match utility of that pair is derived by the expected minimum of $n^2$ draws and is given by $\frac{1}{n^2}$. Conditioning on values exceeding the value for that first pair, utilities are still governed by the same exponential distribution. The next pair matched in the recursive process would then generate an expected match utility that is derived by the minimum of $(n-1)^2$ draws of an exponential variable with parameter 1, conditional on all of the draws being greater than the value for the first pair, and, by iterated expectations, is given by $\frac{1}{n^2}+\frac{1}{(n-1)^2}$. Continuing recursively, we get that the average efficiency of person under the stable matching is given by:

$$\begin{aligned}\frac{S_n}{2n} &= -\frac{1}{n}\left[\frac{1}{n^2}+\left(\frac{1}{n^2}+\frac{1}{(n-1)^2}\right)+\left(\frac{1}{n^2}+\frac{1}{(n-1)^2}+\frac{1}{(n-2)^2}\right)+\cdots\right] \\ &= -\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}.\end{aligned}$$

Since $\log(n+1) < \sum_{k=1}^{n}\frac{1}{k} < \log n + 1$, we have $\lim_{n\to\infty}\frac{S_n}{2n} = 0$.

# 4 Allowing for Unemployment

Throughout the paper we maintain the assumption that all participants prefer to be matched with anyone over remaining unmatched. Our analysis for fully aligned or assortative preferences remains intact even if individuals find some potential partners unacceptable. For instance, consider fully aligned preferences for which match utilities $u_{ij}$ are independently drawn from a continuous distribution over $[a,b]$, where $a<0$, $b>0$, and partners are deemed acceptable whenever they generate a match utility of at least 0. For any realized market, we can repeat the recursive construction of the stable matching of Section 3.3, matching in sequence pairs that generate the highest match utility from all pairs that have not been matched already, provided their match utility is positive.[2] As before, we denote by $S_n$ the aggregate efficiency of the stable matching in such a market. Now, suppose that all agents are acceptable, or alternatively, that partners are deemed acceptable whenever they generate a match utility of at least $a$. Denote the corresponding aggregate efficiency of stable matchings in these markets by $\tilde{S}_n$. Certainly, $\tilde{S}_n \leq S_n$, and Proposition 1 in the body of the paper implies that $\lim_{n\to\infty}\left(b-\frac{S_n}{2n}\right) \leq \lim_{n\to\infty}\left(b-\frac{\tilde{S}_n}{2n}\right) = 0$. Similar arguments can be used for unacceptability with fully assortative preferences.

Allowing unacceptability in markets involving idiosyncratic shocks involves far more subtle

[2] When participants can be viewed as unacceptable, stable matchings satisfy *two* restrictions: 1. There is no blocking pair; and 2. No individual prefers to remain unmatched over matching with their allotted partner.

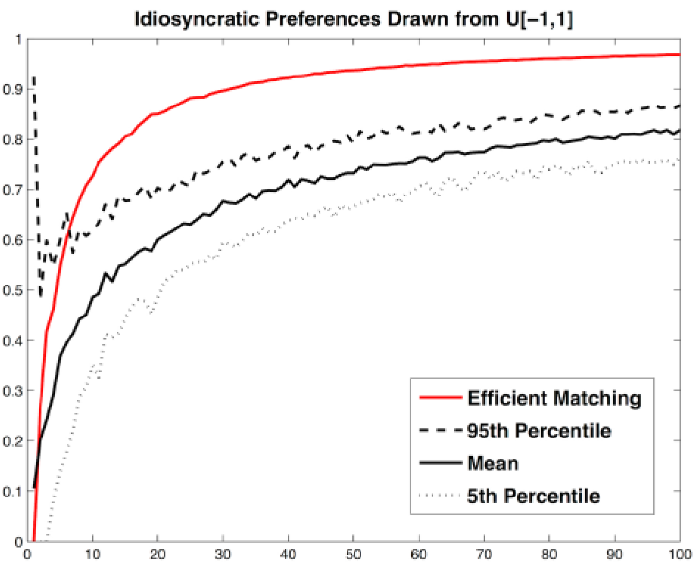


Figure 1: Average Efficiency when Preferences are Idiosyncratic and Individuals May Remain Unmatched

considerations and cannot be translated into our proof methods. Nonetheless, in order to get a sense of the impact of agents finding partners unacceptable, we simulate markets of size $1-100$ with fully idiosyncratic preferences. Match utilities $u_{ij}^f, u_{ij}^w$ are independently drawn from the uniform distribution on $[-1,1]$ and each participant finds a partner acceptable if they generate a utility of at least 0. As before, for each market size, we run 100 simulations, each corresponding to one realization of preferences. For each simulation, we compute the average per-participant utility induced by agents' least preferred stable match partners. In Figure 1 here, the solid black line, the long dashed line, and the short dashed line depict, respectively, the mean, the 95'th percentile, and the 5'th percentile of the simulated distribution of these averages across the 100 simulations, whereas the solid red line corresponds to the maximal average efficiency. Despite focusing on the worst stable partner for all individuals, average per-person utility of the stable matching surpasses 0.8 when market size is 100 (whereas the maximal average efficiency reaches approximately 0.97). The simulation result is comparable to the case in which utilities were drawn from a uniform distribution on $[0,1]$.