Online Appenix of
"Task Allocation and On-the-job Training"

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Abstract

In this Online Appendix we review the results from the queuing theory literature that are relevant to our analysis. We also present the proof of Lemma A1 and provide comparative statics on the welfare impacts of centralization.

1 Primer on Queueing

The limited-monitoring case in Section 4 employs what is termed the M/M/1 queue in the queueing literature, while the perfect-monitoring case in Section 3 employs an M/M/1/k queue. Here we provide a summary of the results relevant to our analysis. For more details, see, for example, Leon-Garcia (2008).

1.1 M/M/1 Queue

Clients seeking service arrive at the market over time $t \in [0, \infty)$ according to a Poisson process with arrival rate ρ. One provider can serve at most one client at any given time. Service completion times are independent across clients and follow an exponential distribution with parameter μ. Upon arrival, each client joins a queue until the provider becomes available. If the provider is not busy helping other clients, the client is served immediately. There is no limit on the possible length of the clients' queue. Clients are served according to a first-in-first-out (FIFO) protocol.

The total number of clients in the system, either waiting in the queue or being served, at time t, N_t, is a continuous-time Markov chain and takes values in $\{0, 1, \ldots, \}$. When $N_t = 0$, there are no clients being served or waiting. When $N_t = 1$, the system has only one client who is being served. When $N_t \geq 2$, at least one client is waiting in the queue. The number of clients N_t increases by one when a client arrives, which occurs at a rate ρ. It decreases by one when the service of a client is completed, which occurs at a rate μ. The ratio $\psi \equiv \frac{\rho}{\mu}$ denotes the provider's *utilization*. As long as $\psi < 1$, N_t has a stationary distribution denoted by $\{p_0, p_1, \dots\}$ such that exactly i clients are in the system with probability p_i.[1] The inflow-outflow equalities, known as the global balance equations, are

$$\begin{aligned} \rho p_0 &= \mu p_1, \\ (\rho + \mu) p_j &= \rho p_{j-1} + \mu p_{j+1}, \quad \forall j = 1, 2, \dots \end{aligned}$$

They yield the stationary distribution

$$p_j = (1 - \psi)\psi^j, \quad j = 0, 1, 2, \dots$$

The average number of clients waiting in the queue, excluding the client currently being served, is

$$\mathbf{E}[Q] \equiv \sum_{j=1}^{\infty} (j-1) p_j = \frac{\psi}{1-\psi} - p_0 = \frac{\psi^2}{1-\psi}.$$

Let $\mathbf{E}[W]$ be the average waiting time in the queue. *Little's formula* guarantees that

$$\mathbf{E}[W] = \frac{\mathbf{E}[Q]}{\rho} = \frac{1}{\mu - \rho} - \frac{1}{\mu}.$$

The intuition behind the formula is the following. Take any time interval, say $[s, s+t)$, during which the system is at the steady state. The total time clients spend waiting in the queue is approximately $\rho t \mathbf{E}[W]$, so the average number of clients waiting in the queue at any given time is $\mathbf{E}[Q] = \rho \mathbf{E}[W]$.

1.2 M/M/1/k Queue

An M/M/1/k queue is similar to an M/M/1 queue but constrains the service provider to accommodate up to k clients, with one client being served, and at most $k-1$ clients waiting in the queue. If a client finds k others present upon arrival, she is turned away. As before, clients' arrival follows a Poisson distribution with parameter ρ, and the provider completes each client's service at times following an exponential distribution with parameter μ. Since, by construction,

[1] The restriction $\psi < 1$ is necessary because, if $\psi \to 1$, the average wait time, which we address shortly, diverges.

the length of the queue is bounded, $\psi \equiv \frac{\rho}{\mu}$ need not be lower than 1.

The total number N_t of clients in the system at time t follows a continuous-time Markov chain over $\{0, 1, 2, \ldots, k\}$. The inflow-outflow equalities, which we omit, yield the following restrictions on the stationary distribution:

$$p_j = \psi^j p_0, \quad \forall j = 1, \ldots, k.$$

Since

$$1 = \sum_{j=0}^{k} p_j = p_0 \sum_{j=0}^{k} \psi^j = \begin{cases} p_0(k+1) & \text{if } \psi = 1, \\ p_0 \left(\frac{1-\psi^{k+1}}{1-\psi} \right) & \text{if } \psi \neq 1, \end{cases}$$

the stationary distribution is given by:

$$p_j = \begin{cases} \frac{1}{k+1} & \text{if } \psi = 1, \\ \frac{\psi^j (1-\psi)}{1-\psi^{k+1}} & \text{if } \psi \neq 1. \end{cases} \quad \forall j = 0, 1, \ldots, k$$

Similarly, the average number of clients in the queue is

$$\begin{aligned} \mathbf{E}[Q] &= \sum_{j=1}^{k} (j-1) p_j = \sum_{j=1}^{k} (j-1) \psi^j p_0 \\ &= \begin{cases} \frac{k(k-1)}{2(k+1)} & \text{if } \psi = 1, \\ p_0 \left(\psi^2 + 2\psi^3 + \cdots + (k-1)\psi^k \right) & \text{if } \psi \neq 1. \end{cases} \end{aligned}$$

In particular, if $\psi \neq 1$, the above expression can be written as:

$$\mathbf{E}[Q] = \frac{1}{1-\psi^{k+1}} \left(\frac{\psi^2 - \psi^{k+1}}{1-\psi} - (k-1)\psi^{k+1} \right).$$

Finally, in the steady state, since a new client is turned away only when $N_t = k$, the average number of clients that join the queue over a unit of time is $\rho(1-p_k)$. Let $\mathbf{E}[W]$ denote those clients' average waiting time. By Little's formula, we have $\mathbf{E}[W] = \frac{\mathbf{E}[Q]}{\rho(1-p_k)}$.

2 Proof of Lemma A1

We divide our arguments into two steps. The first step shows that $\mathbf{E}[Q]$ is continuously differentiable at $\phi = 1$, which allows us to focus on the functional form for the case of $\phi \neq 1$, by taking the value at $\phi = 1$ as $\lim_{\phi \to 1} \mathbf{E}[Q]$, and similarly for $\frac{d\mathbf{E}[Q]}{d\phi}$ at $\phi = 1$. The second step shows that $\mathbf{E}[Q]$ is strictly convex.

1. *Step 1: $E[Q]$ is continuously differentiable at $\phi = 1$.*

   To show Step 1, we first present Lemmas A-C below. To ease expositions, we denote $z \equiv a(1-\phi)+1$ and note that $dz/d\phi = -a$, $\lim_{\phi\to 1} z = 1$, and $\lim_{\phi\to 1} \frac{\log z}{1-\phi} = \lim_{\phi\to 1} \frac{a}{z} = a$.

   Lemma A $\mathbf{E}[Q]$ *is continuous at $\phi = 1$.*

   Proof of Lemma A: Using L'Hopital's rule, we obtain

$$\begin{aligned}\lim_{\phi\to 1} \mathbf{E}[Q] &= \frac{1}{a} + \lim_{\phi\to 1} \frac{1}{1-\phi}\left((\phi^2-1)+1+\frac{\log z}{a\log\phi}\right) \\ &= \frac{(a-1)(a-2)}{2a}.\end{aligned}$$

   ■

   Lemma B *Let* $\varepsilon_\phi \equiv \frac{\phi}{\phi-1} - \frac{1}{\log\phi} - \frac{1}{2}$. *Then,* $\lim_{\phi\to 1} \frac{\varepsilon_\phi}{\log\phi} = \frac{1}{12}$.

   Proof of Lemma B: The proof follows directly from repeat applications of L'Hopital's rule. ■

   Lemma C $\lim_{\phi\to 1} \frac{d\mathbf{E}[Q]}{d\phi}$ *exists in* $\mathbb{R}$.

   Proof of Lemma C: For $\phi \neq 1$, we have

$$\begin{aligned}\frac{d\mathbf{E}[Q]}{d\phi} &= \frac{d}{d\phi}\left(\frac{\phi^2}{1-\phi} + \frac{\log z}{a(1-\phi)\log\phi}\right) \\ &= \frac{2\phi-\phi^2}{(1-\phi)^2} + \frac{-a/z}{a(1-\phi)\log\phi} - \frac{\log z}{a(1-\phi)^2(\log\phi)^2}\left(-\log\phi + \frac{1-\phi}{\phi}\right) \\ &= -1 + \frac{1}{(1-\phi)^2} - \frac{1}{z(1-\phi)\log\phi} - \frac{\log z}{a\phi(1-\phi)\log\phi}\left(\frac{3\phi+1}{2(\phi-1)} - \varepsilon_\phi\right).\end{aligned}$$

   It follows from $\lim_{\phi\to 1} \frac{\log z}{1-\phi} = a$ and Lemma B that

$$\lim_{\phi\to 1} \frac{\varepsilon_\phi \log z}{a\phi(1-\phi)\log\phi} = \lim_{\phi\to 1} \frac{1}{a\phi}\frac{\log z}{1-\phi}\frac{\varepsilon_\phi}{\log\phi} = \frac{1}{12}.$$

   Therefore,

$$\lim_{\phi\to 1} \frac{d\mathbf{E}[Q]}{d\phi} = -\frac{11}{12} + \lim_{\phi\to 1}\left(\frac{\log\phi}{1-\phi}\right)^2 \cdot \lim_{\phi\to 1} h(\phi) = -\frac{11}{12} + \lim_{\phi\to 1} h(\phi),$$

where

$$h(\phi) \equiv \frac{1}{(\log \phi)^3} \left[\log \phi - \frac{1-\phi}{z} + \left(\frac{3\phi + 1}{2a\phi} \right) \log z \right].$$

Using L'Hopital's rule repeatedly,

$$\begin{aligned} \lim_{\phi \to 1} h(\phi) &= \lim_{\phi \to 1} \frac{\phi}{3(\log \phi)^2} \left[\frac{1}{\phi} + \frac{1}{z^2} + \frac{-2a}{4a^2\phi^2} \log z + \frac{3\phi + 1}{2a\phi} \frac{(-a)}{z} \right] \\ &= \frac{2a^2 + (3/2)a - 1}{6}. \end{aligned}$$

Therefore, $\lim_{\phi \to 1} \frac{d\mathbf{E}[Q]}{d\phi}$ exists in $\mathbb{R}$, which proves Lemma C. ■

By Lemma A, $\mathbf{E}[Q]$ is continuous at $\phi = 1$. It is also differentiable at every $\phi \neq 1$. The Mean Value Theorem implies that, for any $\phi \neq 1$, there exists $x_\phi \in (1, \phi)$ such that $\frac{\mathbf{E}[Q](\phi) - \mathbf{E}[Q](1)}{\phi - 1} = \frac{d\mathbf{E}[Q](x_\phi)}{d\phi}$. As $\phi \to 1$, we have $x_\phi \to 1$. Hence, by Lemma C, $\frac{d\mathbf{E}[Q](1)}{d\phi} \equiv \lim_{\phi \to 1} \frac{\mathbf{E}[Q](\phi) - \mathbf{E}[Q](1)}{\phi - 1} = \lim_{x_\phi \to 1} \frac{\mathbf{E}[Q](x_\phi)}{d\phi}$ exists in $\mathbb{R}$. That is, $\mathbf{E}[Q]$ is continuously differentiable at $\phi = 1$, which concludes the proof of Step 1.

Step 2: $\mathbf{E}[Q]$ *is strictly convex.*

To show Step 2, we first present the Lemmas D-G below.

Lemma D *The function* $r(x) = \frac{x}{\log(1-x)}$ *is increasing in* x *and strictly convex on* $(-\infty, 0)$ *and* $(0, 1)$.

Proof of Lemma D: Derivating,

$$r'(x) = \frac{1}{\log(1-x)} + \frac{x}{(1-x)\log^2(1-x)} = \frac{1}{\log^2(1-x)} \left(\log(1-x) + \frac{x}{1-x} \right) > 0.^2$$

Derivating again,

$$\begin{aligned} r''(x) &= \left(\frac{1}{\log(1-x)} + \frac{x}{(1-x)\log^2(1-x)} \right)' \\ &= \frac{(2-x)\log(1-x) + 2x}{(1-x)^2 \log^3(1-x)}. \end{aligned}$$

If $x < 0$, we have $(2-x)\log(1-x) + 2x > 0$, so $r''(x) > 0$. If $0 < x < 1$, we have $g(x) \equiv \log(1-x) + \frac{2x}{2-x} < 0$, because $g(0) = 0$ and $g'(x) = \frac{-1}{1-x} + \frac{4}{(2-x)^2} = \frac{(-x^2+4x-4)+4(1-x)}{(1-x)(2-x)^2} < 0$. Thus, $r''(x) > 0$. ■

[2] For any $y \neq 1$, $-\log y < \frac{1}{y} - 1$, which implies that $\log y > \frac{1-y}{y}$. We substitute $1-x$ for y.

Lemma E *The function* $(a(\phi+1)-1)\frac{\phi-1}{\log(a(1-\phi)+1)}$ *is strictly convex on* $[\underline{\phi},1)$ *and* $(1,1+1/a)$.

Proof of Lemma E: Let $r(\phi)=a(\phi+1)-1$ and $g(\phi)=\frac{\phi-1}{\log(a(1-\phi)+1)}$. Note that $a\phi\geq 1$ because of the training constraint $a(1-q)\phi=1$. Thus, $r(\phi)>0$. Moreover, Lemma D implies that g is increasing and strictly convex (using $x=a(\phi-1)$). Therefore,

$$(r(\phi)g(\phi))''=r''(\phi)g(\phi)+2r'(\phi)g'(\phi)+r(\phi)g''(\phi)\geq r(\phi)g''(\phi)>0.$$

■

Lemma F *The function* $r(\phi)=\frac{1}{\log(a(1-\phi)+1)}+\frac{1}{a\log\phi}$ *is strictly convex on* $[\underline{\phi},1)$ *and* $(1,1+1/a)$.

Proof of Lemma F: Recall that $z\equiv a(1-\phi)+1$, and that $z'\equiv\frac{dz}{d\phi}=-a$. The second derivative of $r(\phi)$ follows from

$$\left(\frac{1}{\log\phi}\right)''=-\left(\frac{1}{\phi\log\phi}\right)'=\frac{1}{\phi^2\log^2\phi}+\frac{2}{\phi^2\log^3\phi},\quad\text{and}$$
$$\left(\frac{1}{\log(a(1-\phi)+1)}\right)''=a^2\left(\frac{1}{\log z}\right)''=a^2\left(\frac{1}{z^2\log^2 z}+\frac{2}{z^2\log^3 z}\right).$$

Thus,

$$r''(\phi)=\frac{a^2}{z^2\log^2 z}\left(1+\frac{2}{\log z}\right)+\frac{1}{a\phi^2\log^2\phi}\left(1+\frac{2}{\log\phi}\right)$$
$$\Longrightarrow(\log^2\phi)r''(\phi)=\frac{a^2}{z^2}\left(\frac{\log\phi}{\log z}\right)^2\left(1+\frac{2}{\log z}\right)+\frac{1}{a\phi^2}\left(1+\frac{2}{\log\phi}\right).$$

We make two claims:

Claim 1: $-\frac{\log\phi}{\log z}>1$ *for every* $\phi\neq 1$.

Proof of Claim 1: By the training constraint, $a(1-q)\phi=1$, so $a\phi\geq 1$. Hence, if $\phi>1$, then $(a\phi)(1-\phi)>(1-\phi)$ and $\phi>\frac{1}{a(1-\phi)+1}$. If $\phi<1$, then $(1-\phi)>(a\phi)(1-\phi)$, so $\frac{1}{\phi}>a(1-\phi)+1$.

Claim 2: $\lim_{\phi\to 1}-\frac{\log z}{\log\phi}=a$ *and is increasing in* ϕ.

Proof of Claim 2: First, $\lim_{\phi\to 1} -\frac{\log z}{\log \phi} = \lim_{\phi\to 1} \frac{a/(a(1-\phi)+1)}{1/\phi} = a$. For any $\phi \neq 1$, define

$$h(\phi) \equiv \phi \left(-\frac{\log z}{\log \phi}\right)' = \frac{a\phi \log \phi}{a(1-\phi)+1} + \log(a(1-\phi)+1).$$

Then,

$$h'(\phi) = \frac{a \log \phi}{a(1-\phi)+1} + \frac{a^2 \phi \log \phi}{(a(1-\phi)+1)^2},$$

which is strictly negative for $\phi < 1$ and strictly positive for $\phi > 1$. Thus, for any $\phi \neq 1$, $h(\phi) > \lim_{\phi \to 1} h(\phi) = 0$, which concludes the proof of Claim 2.

If $\phi \in [\underline{\phi}, 1)$, then $\log \phi < 0$, $\log z > 0$, and

$$\begin{aligned}(\log^2 \phi)(\log x) r''(\phi) &= \frac{a^2}{z^2}\left(\frac{\log \phi}{\log z}\right)^2 (\log z + 2) + \frac{1}{a\phi^2}\left(\log z + \frac{2\log z}{\log \phi}\right)\\ &> \left(\frac{a^2}{z^2} + \frac{1}{a\phi^2}\right)\log x + 2\left(\frac{a^2}{z^2} + \frac{1}{a\phi^2}\frac{\log z}{\log \phi}\right)\\ &> \left(\frac{a^2}{z^2} + \frac{1}{a\phi^2}\right)\log x + 2\left(\frac{a^2}{z^2} - \frac{1}{\phi^2}\right),\end{aligned}$$

where the first inequality follows from Claim 1 and the second from Claim 2. Since $\phi \geq \underline{\phi} = \frac{1+\sqrt{1+4a}}{2a}$ implies $(a\phi)^2 - z^2 = (a\phi)^2 - (a(1-\phi)+1)^2 = (2a\phi - a - 1)(a+1) > 0$, we obtain $r''(\phi) > 0$.

If $\phi \in (1, 1+1/a)$, then $\log \phi > 0$, $\log z < 0$, and

$$(\log^2 \phi)\phi^2 r''(\phi) = \left(\frac{a\phi \log \phi}{z(-\log z)}\right)^2 \left(1 + \frac{2}{\log z}\right) + \frac{1}{a}\left(1 + \frac{2}{\log \phi}\right).$$

Suppose that $1 + \frac{2}{\log z} < 0$, as for otherwise it is clear that $r''(\phi) > 0$. It must be that $a\phi \log \phi + z \log z > 0$ since the left-hand-side is zero at $\phi = 1$, and the derivative $a(\log \phi + 1) - a(\log z + 1) = a(\log \phi - \log z) > 0$. Thus, by Claim 1 above,

$$(\log^2 \phi)\phi^2 r''(\phi) > \left(1 + \frac{2}{\log z}\right) + \frac{1}{a}\left(1 + \frac{2}{\log \phi}\right) = 1 + \frac{1}{a} + 2\left(\frac{1}{\log z} + \frac{1}{a \log \phi}\right) > 0,$$

and $r''(\phi) > 0$, which concludes the proof of Lemma F. ∎

Lemma G *Suppose that $r : \mathbb{R} \to \mathbb{R}$, is positive and decreasing in x and satisfies $g(x)r(x) = h(x)$, with $h(x)$ strictly convex and $g(x)$ strictly positive, concave, and decreasing in x. Then, $r(x)$ is strictly convex.*

Proof of Lemma G: Take any $x_1, x_2 \in \mathbb{R}$ and $\bar{x} = \beta x_1 + (1-\beta)x_2$ for some $\beta \in (0,1)$. Then,

$$\begin{aligned}
&g(\bar{x}) \cdot (\beta r(x_1) + (1-\beta)r(x_2)) \\
&\geq (\beta g(x_1) + (1-\beta)g(x_2)) \cdot (\beta r(x_1) + (1-\beta)r(x_2)) \\
&= \beta^2 g(x_1)r(x_1) + (1-\beta)^2 g(x_2)r(x_2) + \beta(1-\beta)(g(x_1)r(x_2) + g(x_2)r(x_1)),
\end{aligned}$$

where the first inequality is guaranteed by the concavity of g. Since $r(x)$ is increasing and $g(x)$ is decreasing in x,

$$g(x_1)r(x_2) + g(x_2)r(x_1) \geq g(x_1)r(x_1) + g(x_2)r(x_2).$$

$$\begin{aligned}
&\implies g(\bar{x}) \cdot (\beta r(x_1) + (1-\beta)r(x_2)) \geq \beta g(x_1)r(x_1) + (1-\beta)g(x_2)r(x_2) \\
&= \beta h(x_1) + (1-\beta)h(x_2) > h(\bar{x}) = g(\bar{x})r(\bar{x}),
\end{aligned}$$

which implies $\beta r(x_1) + (1-\beta)r(x_2) > r(\bar{x})$. ■

We are now ready to show that

$$\mathbf{E}[Q] = \frac{1}{a} + \frac{1}{1-\phi}\left(\phi^2 + \frac{\log(a(1-\phi)+1)}{a\log\phi}\right),$$

where the value at $\phi = 1$ is given by $\lim_{\phi\to 1}\mathbf{E}[Q]$ is strictly convex in ϕ.

For any $\phi \in [\underline{\phi}, 1) \cup (1, 1+1/a)$, we have

$$\left(\frac{1-\phi}{\log(a(1-\phi)+1)}\right)\mathbf{E}[Q] = (a(\phi+1)-1)\frac{\phi-1}{a\log(a(1-\phi)+1)} + \left(\frac{1}{\log(a(1-\phi)+1)} + \frac{1}{a\log\phi}\right).$$

By Lemma D, $\frac{1-\phi}{\log(a(1-\phi)+1)}$ is decreasing and concave. By Lemmas E and F, the right-hand side is strictly convex. It follows from Lemma G that $\mathbf{E}[Q]$ is strictly convex on $[\underline{\phi}, 1)$ and $(1, 1+1/a)$. Last, by Lemmas A and C, we know that $\mathbf{E}[Q]$ is continuously differentiable at $\phi = 1$, which completes the proof of Step 2. ■

3 Impacts of Centralization: Comparative Statics

We now describe the impact of some parameters of our limited-monitoring case on clients' expected welfare. The average welfare per client can be written as:

$$V = q\,(h - c\mathbf{E}[W]) + (1-q)l = l + q(h-l) - qc\mathbf{E}[W].$$

Denote by V_L^e and V_L^* the average utility per client under the discretionary equilibrium and under the optimal policy, respectively. In the discretionary setting, since in equilibrium clients are indifferent between junior and senior service, $V_L^e = l$. In particular, the welfare gap, $V_L^* - V_L^e$, exhibits the same comparative statics as those of the welfare generated by the socially optimal protocol, V_L^*.

Suppose the value for senior service increases from h_1 to h_2, $h_2 > h_1$, while all other parameters stay fixed. The planner can certainly emulate whatever optimal policy she was following when the value from senior service was h_1. This would yield the same expected waiting costs but increase service quality. Thus, V_L^*, and thereby $V_L^* - V_L^e$, increase in h. A similar argument holds for an increase in the waiting cost c.

The impacts of an increase in arrival rates is more subtle. More rapid arrivals yield more opportunities for training, but also generate more congestion. In general, the effects of increases in λ could go either way. However, for linear training technologies, Proposition 4 in the main text indicates that wait times decrease, implying that all clients served by seniors are better off, and that the optimal fraction of clients served by seniors increases. Consequently, V_L^*, and thus $V_L^* - V_L^e$, increase in λ. Similar comparative statics follow for the training efficacy. We therefore have the following corollary.

Corollary (Welfare Gap Comparative Statics) *Suppose $f(x) = ax$, for some $a > 0$. The relative welfare gain from centralization, $V_L^* - V_L^e$, is increasing in both λ and a.*

Our discussion above considers the average welfare. One may also wish to consider the volume of clients served, thereby focusing on $\lambda\,(V_L^* - V_L^e)$. The comparative statics of Corollary 1 would continue to hold. However, as arrival rates increase, the benefits of centralization would become even more pronounced as more clients are impacted.

This discussion implies that, with limited monitoring, organizations obtain greater advantages from centralization when the quality of senior service improves, when waiting costs decrease, or when either the arrival rate or the training technology efficacy increase.

4 References

Leon-Garcia, Alberto, 2008, Probability, Statistics, and Random Process for Electrical Engineering, Third Edition, Pearson Education, Inc.